Exact solitary waves of a nonlinear network equation

Wen Zhang, Yanzhao Huang, and Yi Xiao*
Physics Department, Huazhong University of Science and Technology, Wuhan 430074, People's Republic of China

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We show by using the real exponential approach that the well-known discrete modified Korteweg–de Vries equation and nonlinear network equations have more general exact soliton solutions than the known bright soliton and kink solutions. Depending on the values of the parameters, the new solutions can describe a series of bright solitons, dark solitons, and kink solitons. [S1063-651X(98)12406-6]

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Recently much work have been done to investigate the effects of discreteness on the dynamics and physical properties of solitons [1,2]. In these investigations, integrable nonlinear lattice models (Toda lattice, lattice nonlinear Schrödinger equation, and so on) have played an important role since it is easy to obtain analytical results. In the present paper we consider a nonlinear electrical lattice, i.e., the nonlinear self-dual network. The nonlinear self-dual network is one of the typical integrable nonlinear lattice systems and it describes the propagation of electrical signals in a cascade of four-terminal nonlinear LC self-dual circuits (see Fig. 1) [3]. The nonlinear self-dual network equations read

$$\dot{V}_j=(1+\gamma V_j^2)(I_j-I_{j+1}), \tag{1}$$

$$\dot{I}_j=(1+\gamma I_j^2)(V_{j-1}-V_j), \tag{2}$$

where $\gamma=\pm 1$, and V_j and I_j are the voltage and current in the jth capacitance and inductance of the network, respectively. The network equations are integrable. By using the inverse scattering method, the Hirota method, the Backlund transformation, and so on, we can show that they have the standard bright soliton solutions:

$$V_j=\sinh(K/2)/\cosh[K(j-j_0)-\omega t], \tag{3}$$

$$I_j=-\sinh(K/2)/\cosh[K(j-j_0-1/2)-\omega t], \tag{4}$$

$$\omega=-2\sinh(K/2)$$

for $\gamma=+1$ and kink solutions

$$V_j=\tanh(K/4)\tanh[(K/2)(j-j_0)-\omega t], \tag{5}$$

$$I_j=-\tanh(K/4)\tanh[(K/2)(j-j_0-1)-\omega t], \tag{6}$$

$$\omega=-2\tanh(K/4)$$

for $\gamma=-1$. In the present paper we show by using the real exponential method that the nonlinear self-dual network equations have more general exact soliton solutions that include not only the known standard bright solitons and kinks but also a series of new superbright, gray, black, and other types of solitary waves, which, to our knowledge, have not been given by other methods.

The real exponential method to find the soliton solutions of nonlinear evolution and wave equations has been proposed by Korpel [4] and developed by Hereman *et al.* [5]. Recently we have successfully applied the real exponential method to a number of integrable and nonintegrable discrete nonlinear evolution and wave equations and found more general or new solitary wave solutions of these equations [7]. So in the present paper we use the real exponential method to investigate the nonlinear self-dual network.

The form of the nonlinear self-dual network is similar to the discrete modified Korteweg–de Vries (DMKdV) equation [8]. In fact, we will show that the soliton solutions of the nonlinear self-dual network have the same functional forms as those of the DMKdV equation. So we first consider the relatively simple DMKdV equation.

(1) *DMKdV equation*. The DMKdV equation has the form [8]

$$\dot{u}_j=(1+\gamma u_j^2)(u_{j+1}-u_{j-1}), \tag{7}$$

where the dot denotes the derivative with respect to time. The standard solutions of the DMKdV equation are the bright soliton

$$u_j=\sinh(K)\operatorname{sech}[(K(j-j_0)-\omega t)], \tag{8}$$

$$\omega=-2\sinh(K)$$

for $\gamma=+1$ and kink soliton [6]

$$u_j=\tanh(K/2)\tanh[(K/2)(j-j_0)-\omega t], \tag{9}$$

$$\omega=-2\tanh(K/2)$$

for $\gamma=-1$. We will show that these solutions are the special cases of more general soliton solutions. In the real exponential approach, the solution of a nonlinear equation is represented as a series in the real exponential solution of the linearized equation [4–7], i.e., we expand u_j in a power series of the form

$$u_j=\sum_{n=0}^{\infty} a_n g_j^n, \tag{10}$$

*Author to whom correspondence should be addressed.

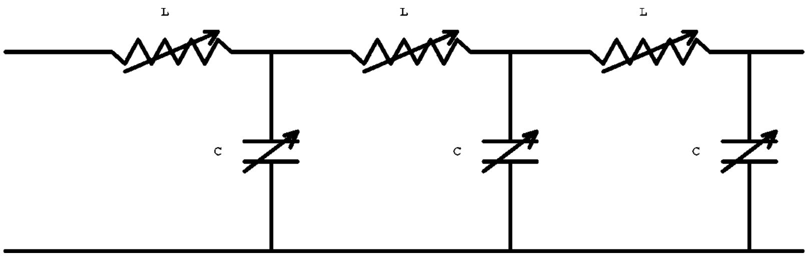

FIG. 1. A nonlinear self-dual network with nonlinear inductance L and capacitance C given by $L(I_j)=I_j^{-1}\tan^{-1}(I_j)$ and $C(V_j)=V_j^{-1}\tan^{-1}(V_j)$ or $L(I_j)=I_j^{-1}\tanh^{-1}(I_j)$ and $C(V_j)=V_j^{-1}\tanh^{-1}(V_j)$, respectively.

where a_n is the expansion coefficient and

$$g_j=\exp[-K(j-j_0)+\omega t], \quad K>0$$

with K, ω, and j_0 being constants. Our task now is to find the analytical form of a_n. This can be done by using symbolic manipulation programs (MATHEMATICA, MAPLE, REDUCE and the like). In the following, we study the DMKdV equation for $\gamma=+1$ and $\gamma=-1$, respectively.

(i) $\gamma=+1$. In this case, the DMKdV equation becomes

$$\dot{u}_j=(1+u_j^2)(u_{j+1}-u_{j-1}). \tag{11}$$

Substituting Eq. (10) into Eq. (11) and equating the coefficients of g_j^n, we find

$$\omega=-2(1+a_0^2)\sinh(K), \tag{12}$$

$$a_1=\text{arbitrary constant},$$

$$a_2=-\frac{2sa_1^2}{2(1+a_0^2)\sinh(K)},$$

$$a_3=\frac{(3s^2-1)a_1^3}{[2(1+a_0^2)\sinh(K)]^2},$$

$$a_4=-\frac{4s(s^2-1)a_1^4}{[2(1+a_0^2)\sinh(K)]^3},$$

$$a_5=\frac{(1-10s^2+5s^4)a_1^5}{[2(1+a_0^2)\sinh(K)]^4},$$

etc., where $s=a_0/\tanh(K/2)$. It is not easy to see directly from the sequence above whether a_n has a common expression but we found after some tries that it does have one that can be written as

$$a_n=\frac{(s+I)^n-(s-I)^n}{2I}\frac{a_1^n}{[-2(1+a_0^2)\sinh(K)]^{n-1}}, \tag{13}$$

where $I=\sqrt{-1}$ is the imaginary unit. Having the common expression of a_n, we can obtain a closed form of u_j from Eq. (10):

$$u_j=u_\infty+\frac{2(1+u_\infty^2)\sinh(K)}{2u_\infty\coth(K/2)+\{[1+u_\infty^2\coth^2(K/2)]g_j+g_j^{-1}\}}, \tag{14}$$

where $u_\infty=a_0$ is the boundary value of u_j when j approaches $+\infty$ and a_1 has been chosen to be

$$a_1=-2(1+u_\infty^2)\sinh(K).$$

It is easy to check that Eq. (14) is an exact solution of Eq. (11). It is noted that, if u_j is a solution of Eq. (11), so is $-u_j$.

Equation (14) is a general solution of the DMKdV equation (11) and the known bright soliton solution is its special case when the boundary value $u_\infty=0$. In the general case, the solution (14) has two controlling parameters u_∞ and K while the known bright solitary wave solution has only one. The general solution can represent soliton solutions with different boundary (background) values u_∞ and their widths and heights become narrow and large, respectively, as the values of u_∞ go from positive to negative for given K. This can be clearly seen from Fig. 2.

(ii) $\gamma=-1$. In this case, Eq. (7) becomes

$$\dot{u}_j=(1-u_j^2)(u_{j+1}-u_{j-1}). \tag{15}$$

Substituting Eq. (10) into Eq. (11) and equating the coefficients of g_j^n, we find

$$\begin{aligned}\omega&=-2(1-a_0^2)\sinh(K),\\ a_1&=\text{arbitrary constant},\\ a_2&=\frac{2sa_1^2}{2(1-a_0^2)\sinh(K)},\\ a_3&=-\frac{(1+3s^2)a_1^3}{[2(1-a_0^2)\sinh(K)]^2},\\ a_4&=\frac{4s(1+s^2)a_1^4}{[2(1-a_0^2)\sinh(K)]^3},\\ a_5&=-\frac{(1+10s^2+3s^4)a_1^5}{[2(1-a_0^2)\sinh(K)]^4},\end{aligned} \tag{16}$$

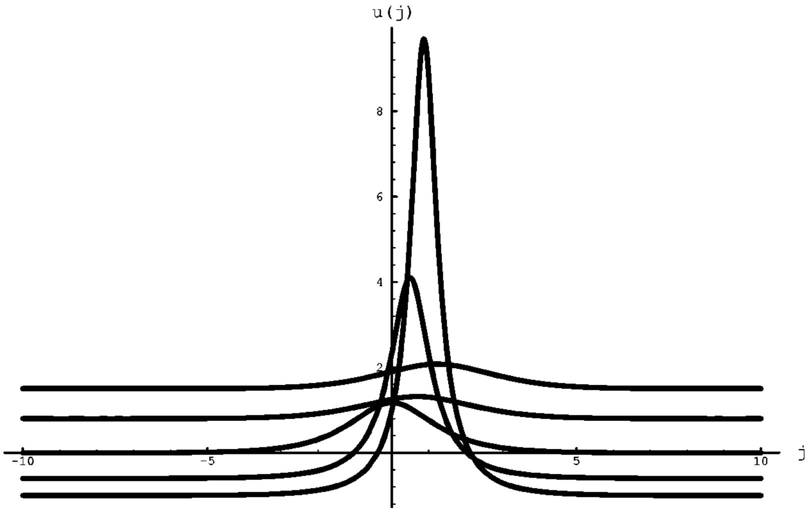

FIG. 2. Shapes u_j of the solitary waves [Eq. (14)] of the DMKdV equation with $K=1.0$ and different boundary values (from top to bottom): $u_\infty=1.5$, 0.8, 0.0, -0.6, and -1.0.

etc., where $s=a_0/\tanh(K/2)$. We find that the common expression of a_n can be written as in this case

$$a_n=\frac{(s+1)^n-(s-1)^n}{2}\frac{(-a_1)^n}{[2(1-a_0^2)\sinh(K)]^{n-1}} \tag{17}$$

Then we find from Eq. (10)

$$u_j=u_\infty-\frac{2(1-u_\infty^2)\sinh(K)}{2u_\infty\coth(K/2)+\{[u_\infty^2\coth^2(K/2)-1]g_j+g_j^{-1}\}} \tag{18}$$

where $u_\infty=a_0$ and a_1 has been chosen to be

$$a_1=2(1-u_\infty^2)\sinh(K).$$

Equation (18) is a general solution of the DMKdV equation (15) and the known kink solution can be obtained by setting $u_\infty=\tanh(K/2)$. In the general case, the general solution can represent bright and dark soliton solutions with different boundary (background) values. This can be clearly seen from Fig. 3. When $u_\infty>\tanh(K/2)$, Eq. (18) gives kink-antikink pairlike solitary waves, black solitary wave (dark solitons with the value of the bottom being zero), gray solitary waves (hole solitons), and superbright solitary waves (bright solitary waves on positive backgrounds).

(2) *Nonlinear Self-Dual Network*. Having obtained the solutions of the DMKdV equation, we consider the nonlinear self-dual network equation (1). Now we must expand both V_j and I_j in power series of the form

$$V_j=\sum_{n=0}^{\infty}a_n g_j^n, \tag{19}$$

$$I_j=\sum_{n=0}^{\infty}b_n g_j^n, \tag{20}$$

where a_n and b_n are the expansion coefficients. In the following, we set $a_0=-b_0$ and $s=a_0/\tanh(K/4)$.

(i) $\gamma=+1$. In the same way as for the DMKdV equation, we find the expansion coefficients a_n and b_n to be

$$\omega=-2(1+a_0^2)\sinh(K/2), \tag{21}$$

$$a_n=\frac{(s+I)^n-(s-I)^n}{2I}\frac{a_1^n}{[-2(1+a_0^2)\sinh(K/2)]^{n-1}}, \tag{22}$$

$$b_n=-\exp(nK/2)a_n. \tag{23}$$

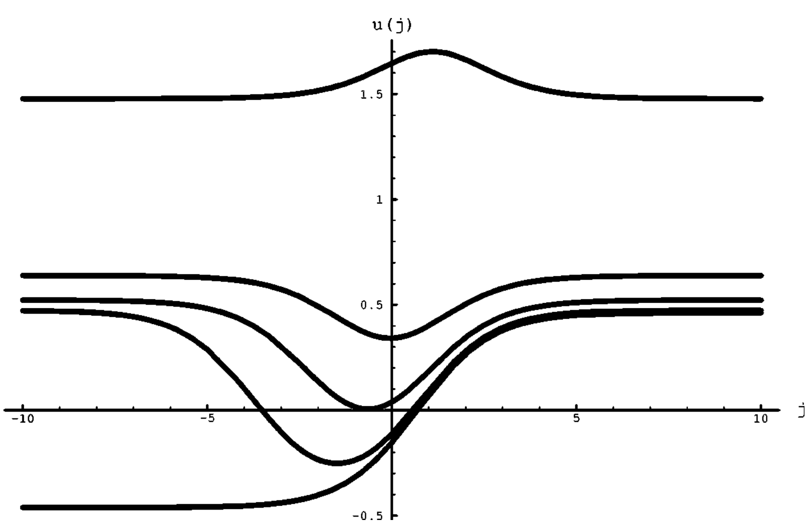

FIG. 3. Shapes u_j of the solitary waves [Eq. (18)] of the DMKdV equation with $K=1.0$ and different boundary values when j approaches $+\infty$ (from top to bottom): $u_\infty=1.478$ (super-bright soliton), 0.693 (gray soliton), 0.522 (black soliton), 0.474 (kink-antikink-pairlike soliton) and tanh(1.0) (kink).

It is clear that the form of a_n is the same as Eq. (13) except that K is replaced by $K/2$. From Eqs. (19) and (20) we find

$$V_j=a_0+\frac{2(1+a_0^2)\sinh(K/2)}{2a_0\coth(K/4)+\{[1+a_0^2\coth^2(K/4)]g_j+g_j^{-1}\}}, \tag{24}$$

$$I_j=-a_0-\frac{2(1+a_0^2)\sinh(K/2)}{2a_0\coth(K/4)+\{[1+a_0^2\coth^2(K/4)]g_je^{K/2}+g_j^{-1}e^{-K/2}\}}. \tag{25}$$

The solutions (24) and (25) have the same form as the solution (14) of the dark DMKdV equation and so also represent different kinds of solitary waves (Fig. 2). It is also clear that the V_j has the same functional form as I_j except a position shift by $K/2$. For $a_0=0$, we obtain the known bright soliton solution (3) and (4).

(ii) $\gamma=-1$. Similarly, we find the expansion coefficients a_n and b_n to be

$$\omega=-2(1-a_0^2)\sinh(K/2), \tag{26}$$

$$a_n=\frac{(s+1)^n-(s-1)^n}{2}\frac{a_1^n}{[2(1-a_0^2)\sinh(K/2)]^{n-1}}, \tag{27}$$

$$b_n=-\exp(nK/2)a_n. \tag{28}$$

It is also clear that the form of a_n is the same as Eq. (17) except that K is replaced by $K/2$. Then, we find

$$V_j=a_0-\frac{2(1-a_0^2)\sinh(K/2)}{2a_0\coth(K/4)+\{[a_0^2\coth^2(K/4)-1]g_j+g_j^{-1}\}}, \tag{29}$$

$$I_j=-a_0+\frac{2(1-a_0^2)\sinh(K/2)}{2a_0\coth(K/4)+\{[a_0^2\coth^2(K/4)-1]g_je^{K/2}+g_j^{-1}e^{-K/2}\}}. \tag{30}$$

The solutions (29) and (30) have the same form as the solutions of Eq. (15) and so also represent superbright solitary waves, gray solitary waves, black solitary wave, and kink-antikink pairlike solitary waves and kink solitons (Fig. 3). When $a_0=\tanh(K/4)$, we obtain the kink solitons (5) and (6).

The results above show that the nonlinear self-dual networks can support the richness of nonlinear waves, which include various bright and dark solitary waves. Nonlinear electrical lattices are very convenient tools to study the wave propagation in a one-dimensional nonlinear dispersive system and have been used to study the properties of solitons on lattices [9,2]. They also have a range of applications, including their use in sampling oscilloscopes, network analyzers, pulse compression devices, and so on [10]. We hope that the solitary wave solutions given in the present paper can have some practical applications.

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